

TWO DIVISORS OF $(n^2 + 1)/2$ SUMMING UP TO $\delta n + \varepsilon$, FOR δ AND ε EVEN

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ABSTRACT. In this paper we are dealing with the problem of the existence of two divisors of $(n^2 + 1)/2$ whose sum is equal to $\delta n + \varepsilon$, in the case when δ and ε are even, or more precisely in the case in which $\delta \equiv \varepsilon + 2 \equiv 0$ or $2 \pmod{4}$. We will completely solve the cases $\delta = 2, \delta = 4$ and $\varepsilon = 0$.

1. INTRODUCTION

In [1], Ayad and Luca have proved that there does not exist an odd integer $n > 1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = n + 1$. In [2], Dujella and Luca have dealt with a more general issue, where $n + 1$ was replaced with an arbitrary linear polynomial $\delta n + \varepsilon$, where $\delta > 0$ and ε are given integers. The reason that d_1 and d_2 are congruent to 1 modulo 4 comes from the fact that $(n^2 + 1)/2$ is odd and is a sum of two coprime squares $((n + 1)/2)^2 + ((n - 1)/2)^2$. Such numbers have the property that all their prime factors are congruent to 1 modulo 4. Since $d_1 + d_2 = \delta n + \varepsilon$, then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod{2}$, or $\delta \equiv \varepsilon + 2 \equiv 0$ or $2 \pmod{4}$. In [2] authors have focused on the first case.

In this paper, we deal with the second case, the case where $\delta \equiv \varepsilon + 2 \equiv 0$ or $2 \pmod{4}$. We completely solve cases when $\delta = 2, \delta = 4$ and $\varepsilon = 0$. We prove that there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$ for $\varepsilon \equiv 0 \pmod{4}$ and we prove an analogous result for $\varepsilon \equiv 2 \pmod{4}$ and divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$. In case when $\delta \geq 6$ is a positive integer of the form $\delta = 4k + 2$, $k \in \mathbb{N}$ we prove that there does not exist an odd integer n such that there exists a pair of divisors d_1, d_2 of $\frac{n^2+1}{2}$ with the property $d_1 + d_2 = \delta n$. We also prove that there exist infinitely many odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$.

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2. THE CASE $\delta = 2$

Theorem 1. *If $\varepsilon \equiv 0 \pmod{4}$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.*

Proof. Let $\varepsilon \equiv 0 \pmod{4}$. We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$. Let $g = \gcd(d_1, d_2)$. We can write $d_1 = gd'_1, d_2 = gd'_2$. Since $gd'_1d'_2 = \text{lcm}(d_1, d_2)$ divides $\frac{n^2+1}{2}$, we conclude that there exists a positive integer d such that

$$d_1d_2 = \frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1d_2,$$

we can easily obtain

$$\begin{aligned} (d_2 - d_1)^2 &= (2n + \varepsilon)^2 - 4\frac{g(n^2+1)}{2d}, \\ (d_2 - d_1)^2 &= 4n^2 + 4\varepsilon n + \varepsilon^2 - 2\frac{g(n^2+1)}{d}, \\ d(d_2 - d_1)^2 &= 4n^2d + 4d\varepsilon n + \varepsilon^2d - 2n^2g - 2g, \\ d(d_2 - d_1)^2 &= (4d - 2g)n^2 + 4d\varepsilon n + \varepsilon^2d - 2g, \\ d(4d - 2g)(d_2 - d_1)^2 &= (4d - 2g)^2n^2 + 4(4d - 2g)d\varepsilon n + 4d^2\varepsilon^2 - 8dg - 2\varepsilon^2dg + 4g^2. \end{aligned} \tag{2.1}$$

For $X = (4d - 2g)n + 2d\varepsilon, Y = d_2 - d_1$, the equation (2.1) becomes

$$X^2 - d(4d - 2g)Y^2 = 8dg + 2\varepsilon^2dg - 4g^2.$$

For $g = 1$ the previous equation becomes

$$\begin{aligned} X^2 - 2d(2d - 1)Y^2 &= 8d + 2\varepsilon^2d - 4, \\ X^2 - 2d(2d - 1)Y^2 &= 2d(4 + \varepsilon^2) - 4. \end{aligned} \tag{2.2}$$

The equation (2.2) is a Pellian equation. The right-hand side of (2.2) is nonzero.

Our goal is to make the right-hand side of (2.2) a perfect square. That condition can be satisfied by taking $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$. With this choice of d , we get

$$2d(4 + \varepsilon^2) - 4 = 2\left(\frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1\right)(4 + \varepsilon^2) - 4 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2.$$

Pellian equation (2.2) becomes

$$X^2 - 2d(2d - 1)Y^2 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2. \quad (2.3)$$

Now, like in [2], we are trying to solve (2.3). We let

$$X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U, \quad Y = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)V.$$

The equation (2.3) becomes

$$U^2 - 2d(2d - 1)V^2 = 1. \quad (2.4)$$

Equation (2.4) is a Pell equation which has infinitely many positive integer solutions (U, V) , and consequently, there exist infinitely many positive integer solutions (X, Y) of (2.3). The least positive integer solution of (2.4) can be found using the continued fraction expansion of number $\sqrt{2d(2d - 1)}$.

We can easily get $\sqrt{2d(2d - 1)} = [2d - 1; \overline{2, 4d - 2}]$. All positive solutions of (2.4) are given by (U_m, V_m) for some $m \geq 0$. The first few solutions are

$$\begin{aligned} (U_0, V_0) &= (1, 0), \\ (U_1, V_1) &= (4d - 1, 2), \\ (U_2, V_2) &= (32d^2 - 16d + 1, 16d - 4), \\ (U_3, V_3) &= (256d^3 - 192d^2 + 36d - 1, 128d^2 - 64d + 6), \dots \end{aligned}$$

Generally, solutions of (2.4) are generated by recursive expressions

$$\begin{aligned} U_0 &= 1, \quad U_1 = 4d - 1, \quad U_{m+2} = 2(4d - 1)U_{m+1} - U_m, \\ V_0 &= 0, \quad V_1 = 2, \quad V_{m+2} = 2(4d - 1)V_{m+1} - V_m, \quad m \in \mathbb{N}_0. \end{aligned} \quad (2.5)$$

By induction on m , one gets that $U_m \equiv 1 \pmod{(4d - 2)}$, $m \geq 0$. Indeed, $U_0 = 1 \equiv 1 \pmod{(4d - 2)}$, $U_1 = 4d - 1 \equiv 1 \pmod{(4d - 2)}$. Assume that $U_m \equiv U_{m-1} \equiv 1 \pmod{(4d - 2)}$. For U_{m+1} we get

$$U_{m+1} = 2(4d - 1)U_m - U_{m-1} \equiv 2 - 1 \equiv 1 \pmod{(4d - 2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X = (4d - 2)n + 2d\varepsilon$ and $X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U$. We obtain

$$n = \frac{\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U - 2d\varepsilon}{4d - 2}.$$

We want the above number n to be a positive integer.

From $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$, it follows $4d - 2 = \frac{1}{2}\varepsilon^2 - 2\varepsilon + 2$. Note that ε is even. So, congruences

$$\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U - 2d\varepsilon \equiv 4d + \varepsilon - 2 - 2d\varepsilon \equiv -(2d - 1)\varepsilon \equiv 0 \pmod{(4d - 2)},$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n , which we get from U_1, U_2, U_3 , are

$$\begin{cases} n = \frac{1}{2}(\varepsilon^2 - 3\varepsilon + 6), \\ d_1 = 1, \\ d_2 = \varepsilon^2 - 2\varepsilon + 5. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^4 - 6\varepsilon^3 + 20\varepsilon^2 - 33\varepsilon + 34), \\ d_1 = \varepsilon^2 - 2\varepsilon + 5, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^6 - 10\varepsilon^5 + 50\varepsilon^4 - 148\varepsilon^3 + 281\varepsilon^2 - 323\varepsilon + 198), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 49\varepsilon^4 - 142\varepsilon^3 + 262\varepsilon^2 - 292\varepsilon + 169. \end{cases}$$

□

3. THE CASE $\delta = 4$

Theorem 2. *If $\varepsilon \equiv 2 \pmod{4}$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.*

Proof. Proof of this theorem will be slightly different from the proof of Theorem 1. Instead of assuming that $\varepsilon \equiv 2 \pmod{4}$, we will distinguish two cases: in one case we will be dealing with $\varepsilon \equiv 6 \pmod{8}$ and we will apply strategies from [2] and in the other case we will be dealing with $\varepsilon \equiv 2 \pmod{8}$ and we will use different methods in obtaining results.

We start with the case when $\varepsilon \equiv 6 \pmod{8}$. We want to find odd positive integers n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.

Let $g = \gcd(d_1, d_2)$, $d_1 = gd'_1$, $d_2 = gd'_2$ and d is a positive integer which satisfies the equation

$$d_1 d_2 = \frac{g(n^2 + 1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we obtain

$$\begin{aligned} (d_2 - d_1)^2 &= (4n + \varepsilon)^2 - 4 \frac{g(n^2 + 1)}{2d}, \\ d(d_2 - d_1)^2 &= (16d - 2g)n^2 + 8d\varepsilon n + \varepsilon^2 d - 2g, \end{aligned}$$

$$d(16d-2g)(d_2-d_1)^2 = (16d-2g)^2 n^2 + 8(16d-2g)d\varepsilon n + 16d^2\varepsilon^2 - 32dg - 2\varepsilon^2 dg + 4g^2. \quad (3.1)$$

Let $X = (16d - 2g)n + 4d\varepsilon$, $Y = d_2 - d_1$. Equation (3.1) becomes

$$X^2 - 2d(8d - g)Y^2 = 32dg + 2\varepsilon^2 dg - 4g^2. \quad (3.2)$$

For $g = 1$ the previous expression becomes

$$X^2 - 2d(8d - 1)Y^2 = 2d(16 + \varepsilon^2) - 4. \quad (3.3)$$

It is obvious that (3.3) is a Pellian equation. The right-hand side of (3.3) is nonzero.

Our goal is to make the right-hand side of (3.3) a perfect square. That condition can be satisfied by taking $d = \frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8}$. With this choice for d , we get

$$2d(16 + \varepsilon^2) - 4 = 2 \left(\frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8} \right) (16 + \varepsilon^2) - 4 = \left(\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16) \right)^2.$$

So, Pellian equation (3.3) becomes

$$X^2 - 2d(8d - 1)Y^2 = \left(\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16) \right)^2. \quad (3.4)$$

Let

$$X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W, \quad Y = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)Z.$$

The equation (3.4) becomes

$$W^2 - 2d(8d - 1)Z^2 = 1. \quad (3.5)$$

The equation (3.5) is a Pell equation which has infinitely many positive integer solutions (W, Z) , and consequently, there exist infinitely many positive integer solutions (X, Y) of (3.4). The least positive integer solution of (3.5) can be found using the continued fraction expansion of number $\sqrt{2d(8d - 1)}$.

We can easily get

$$\sqrt{2d(8d - 1)} = [4d - 1; \overline{1, 2, 1, 8d - 2}].$$

All positive solutions of (3.5) are given by (W_m, Z_m) for some $m \geq 0$. The first few solutions are

$$(W_0, Z_0) = (1, 0),$$

$$(W_1, Z_1) = (16d - 1, 4),$$

$$(W_2, Z_2) = (512d^2 - 64d + 1, 128d - 8), \dots$$

Generally, solutions of (3.5) are generated by recursive expressions

$$W_0 = 1, \quad W_1 = 16d - 1, \quad W_{m+2} = 2(16d - 1)W_{m+1} - W_m,$$

$$Z_0 = 0, \quad Z_1 = 4, \quad Z_{m+2} = 2(16d - 1)Z_{m+1} - Z_m, \quad m \in \mathbb{N}_0.$$

By induction on m , one gets that $W_m \equiv 1 \pmod{(16d-2)}$, $m \geq 0$. Indeed, $W_0 = 1 \equiv 1 \pmod{(16d-2)}$, $W_1 = 16d - 1 \equiv 1 \pmod{(16d-2)}$. Assume that $W_m \equiv W_{m-1} \equiv 1 \pmod{(16d-2)}$. For W_{m+1} we get

$$W_{m+1} = 2(16d-1)W_m - W_{m-1} \equiv 2 - 1 \equiv 1 \pmod{(16d-2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X = (16d-2)n + 4d\varepsilon$ and $X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W$. We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

We want to prove that number n is a positive integer.

From $d = \frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8}$, it follows $8d - 1 = \frac{1}{4}\varepsilon^2 - \varepsilon + 4$. Number $\frac{\varepsilon}{2}$ is an odd integer. Thus, the congruences

$$\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon \equiv 8d - 1 + \frac{\varepsilon}{2} - 4d\varepsilon \equiv (8d - 1)(1 - \frac{\varepsilon}{2}) \equiv 0 \pmod{(16d-2)}$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n , which we get from W_1, W_2, W_3 , are

$$\begin{cases} n = \frac{1}{4}(\varepsilon^2 - 3\varepsilon + 18), \\ d_1 = 1 \\ d_2 = \varepsilon^2 - 2\varepsilon + 17. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^4 - 6\varepsilon^3 + 44\varepsilon^2 - 105\varepsilon + 322), \\ d_1 = \varepsilon^2 - 2\varepsilon + 17, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^6 - 10\varepsilon^5 + 86\varepsilon^4 - 388\varepsilon^3 + 1529\varepsilon^2 - 3155\varepsilon + 5778), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 85\varepsilon^4 - 382\varepsilon^3 + 1486\varepsilon^2 - 3052\varepsilon + 5473. \end{cases}$$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod{8}$. Let $\varepsilon = 8k + 2$, $k \in \mathbb{N}_0$. For $g = \frac{1}{4}\varepsilon^2 + 4$ and $g = d_1$, the equation (3.2) becomes

$$X^2 - 2d(8d - g)Y^2 = \frac{2d-1}{4}\varepsilon^4 + 8\varepsilon^2(2d-1) + 64(2d-1).$$

The right-hand side of the equation will be a perfect square if $2d-1$ is a perfect square. Motivated by the experimental data, we take

$$d = \frac{1}{512}\varepsilon^4 - \frac{1}{64}\varepsilon^3 + \frac{7}{64}\varepsilon^2 - \frac{5}{16}\varepsilon + \frac{41}{32}.$$

We get

$$2d - 1 = 16k^4 + 8k^2 + 1 = (4k^2 + 1)^2.$$

So, the equation (3.2) becomes

$$X^2 - 2d(8d - g)Y^2 = \left(\frac{1}{32}(\varepsilon^2 + 16)(\varepsilon^2 - 4\varepsilon + 20) \right)^2. \quad (3.6)$$

We consider the corresponding Pell equation

$$U^2 - 2d(8d - g)V^2 = 1. \quad (3.7)$$

Let (U_0, V_0) be the least positive integer solution of (3.7). That equation has infinitely many solutions. From (3.7) we get that

$$U^2 \equiv 1 \pmod{(16d - 2g)}.$$

We deal with the case where $g = d_1 = \frac{1}{4}\varepsilon^2 + 4$ and from the experimental data we can set

$$d_2 = d_1^2 - 16kd_1, \quad k \in \mathbb{N}_0.$$

For $Y = d_2 - d_1$ we get

$$Y = \left(\frac{1}{4}\varepsilon^2 + 4 \right)^2 - (2\varepsilon - 3) \left(\frac{1}{4}\varepsilon^2 + 4 \right) = \frac{\varepsilon^4}{16} - \frac{\varepsilon^3}{2} + \frac{11\varepsilon^2}{4} - 8\varepsilon + 28.$$

From (3.6), we obtain:

$$X = \frac{(\varepsilon^2 + 16)(\varepsilon^6 - 16\varepsilon^5 + 140\varepsilon^4 - 768\varepsilon^3 + 3120\varepsilon^2 - 8704\varepsilon + 14400)}{2048}.$$

We claim that X satisfies the congruence

$$X \equiv 4d\varepsilon \pmod{(16d - 2g)}. \quad (3.8)$$

Indeed,

$$\begin{aligned} 16d - 2g &= \frac{\varepsilon^4}{32} - \frac{\varepsilon^3}{4} + \frac{5\varepsilon^2}{4} - 5\varepsilon + \frac{25}{2}, \\ X - 4d\varepsilon &= \left(\frac{\varepsilon^4}{32} - \frac{\varepsilon^3}{4} + \frac{5\varepsilon^2}{4} - 5\varepsilon + \frac{25}{2} \right) \left(\frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9 \right). \end{aligned}$$

From $n = \frac{X - 4d\varepsilon}{16d - 2g}$, we get

$$n = \frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9 = 64k^4 + 28k^2 + 7,$$

and we see that n is an odd integer. Thus, if we define

$$\begin{aligned} (X_0, Y_0) &= \left(\frac{(\varepsilon^2 + 16)(\varepsilon^6 - 16\varepsilon^5 + 140\varepsilon^4 - 768\varepsilon^3 + 3120\varepsilon^2 - 8704\varepsilon + 14400)}{2048}, \right. \\ &\quad \left. \frac{1}{16}(\varepsilon^2 + 16)(\varepsilon^2 - 8\varepsilon + 28) \right), \end{aligned}$$

we see that (X_0, Y_0) is a solution of (3.6) which satisfies the congruence (3.8). We have proved that for every $\varepsilon \equiv 2 \pmod{8}$ there exists at least one odd integer n which satisfies the conditions of Theorem 2. Our goal is to prove that there exist infinitely many such integers n that satisfy the properties of Theorem 2.

If (X_0, Y_0) is a solution of (3.6), solutions of (3.6) are also

$$(X_i, Y_i) = \left(X_0 + \sqrt{2d(8d-g)}Y_0 \right) \left(U_0 + \sqrt{2d(8d-g)}V_0 \right)^{2i}, \quad i = 0, 1, 2, \dots \quad (3.9)$$

From the equation (3.9), we get

$$X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 4d\varepsilon \pmod{(16d-2g)}.$$

So, there are infinitely many solutions (X_i, Y_i) of (3.6) that satisfy the congruence (3.8). Therefore, by

$$n = \frac{X_i - 4d\varepsilon}{16d - 2g},$$

we get infinitely many integers n with the required properties. It is easy to see that number n defined in this way is odd. Indeed, we have $16d - 2g \equiv 2 \pmod{4}$, $X_0 \equiv 2 \pmod{4}$, and since (3.7) implies that U_0 is odd and V_0 is even, we get from (3.8) that

$$X_i - 4d\varepsilon \equiv X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 2 \pmod{4},$$

so n is odd. □

4. THE CASE $\varepsilon = 0$

Proposition 1. *There exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. These solutions satisfy $\gcd(d_1, d_2) = 1$ and $d_1 d_2 = \frac{n^2+1}{2}$.*

Proof. We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. Let $g = \gcd(d_1, d_2)$. Then $g|(2n)$ and $g|(n^2 + 1)$ which implies that $g|((2n)^2 + 4)$ so we can conclude that $g|4$. Because g is the greatest common divisor of d_1, d_2 and d_1, d_2 are odd numbers, we can also conclude that g is an odd number. So, $g = 1$. Like we did in the proofs of the previous theorems, we define a positive integer d which satisfies the equation $d_1 d_2 = \frac{n^2+1}{2d}$. From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we can easily obtain

$$(d_2 - d_1)^2 = (2n)^2 - 2\frac{(n^2 + 1)}{d},$$

$$d(d_2 - d_1)^2 = 4n^2d - 2n^2 - 2.$$

Let $d_2 - d_1 = 2y$, so we get

$$(4d - 2)n^2 - 4dy^2 = 2,$$

$$(2d - 1)n^2 - 2dy^2 = 1. \quad (4.1)$$

We will use the next lemma, which is Criterion 1 from [3] to check if there exists a solution for (4.1).

Lemma 1. *Let $a > 1$, b be positive integers such that $\gcd(a, b) = 1$ and $D = ab$ is not a perfect square. Moreover, let (u_0, v_0) denote the least positive integer solution of the Pell equation*

$$u^2 - Dv^2 = 1.$$

Then equation $ax^2 - by^2 = 1$ has a solution in positive integers x, y if and only if

$$2a|(u_0 + 1) \text{ and } 2b|(u_0 - 1).$$

□

We want to solve the Pell equation

$$U^2 - 2d(2d - 1)V^2 = 1, \quad (4.2)$$

where $n = U$, $y = V$. The continued fraction expansion of the number $\sqrt{2d(2d - 1)}$ is already known from Theorem 1 where we have obtained

$$\sqrt{2d(2d - 1)} = [2d - 1; \overline{2, 4d - 2}].$$

The least positive integer solution of the Pell equation (4.2) is $(4d - 1, 2)$. In our case, we want to find solutions of (4.1), so we apply Lemma 1 which gives us conditions that have to be fulfilled. It has to be that

$$2(2d - 1)|4d \text{ and } 4d|(4d - 2),$$

which is not true for $d \in \mathbb{N}$. So, for Pellian equation (4.1) there are no integer solutions (n, y) when $a = 2d - 1 > 1$. Finally, we have to check the remaining case for $a = 1$, which is the case that is not included in Lemma 1.

If $a = 2d - 1 = 1$, then $d = 1$. From (4.1) and $d = 1$, we get the Pell equation

$$n^2 - 2y^2 = 1, \quad (4.3)$$

which has infinitely many solutions $n = U_m$, $y = V_m$, $m \in \mathbb{N}_0$ where

$$U_0 = 1, \quad U_1 = 3, \quad U_{m+2} = 6U_{m+1} - U_m,$$

$$V_0 = 0, \quad V_1 = 2, \quad V_{m+2} = 6V_{m+1} - V_m, \quad m \in \mathbb{N}_0.$$

The first few values (U_i, V_i) are

$$(U_0, V_0) = (1, 0), \quad (U_1, V_1) = (3, 2), \quad (U_2, V_2) = (17, 12), \quad (U_3, V_3) = (99, 70), \dots$$

From those solutions we can easily generate (n, d_1, d_2)

$$(n, d_1, d_2) = (3, 1, 5), \quad (17, 5, 29), \quad (99, 29, 169), \dots$$

We have proved that there exist infinitely many odd positive integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. We have also proved that $g = 1$ and $d = 1$, so we conclude that numbers d_1 and d_2 are coprime and that $d_1 d_2 = \frac{n^2+1}{2}$. \square

Theorem 3. *Let $\delta \geq 6$ be a positive integer such that $\delta = 4k+2, k \in \mathbb{N}$. Then there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.*

Proof. Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k+2, k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. Let $g = \gcd(d_1, d_2) > 1$. Since $d_1 = g d'_1, d_2 = g d'_2$, it follows that $g | (n^2 + 1)$ and $g | (\delta n)$ and we conclude that $g | ((\delta n)^2 + \delta^2)$, which implies that $g | \delta^2$. This means that g and δ have a common prime factor p . Let $d_1 = p d''_1, d_2 = p d''_2, \delta = p \delta''$. Then, we have $p d''_1 + p d''_2 = p \delta'' n$, so we can conclude $d''_1 + d''_2 = \delta'' n$ where d''_1, d''_2 are divisors of $\frac{n^2+1}{2}$. It is clear that $\delta'' < \delta$ and if it also satisfies $\delta'' \neq 2$, the existence of the number δ'' contradicts the minimality of δ . So, if $\delta'' \neq 2$, then we must have $g = 1$.

If $\delta'' = 2$, it follows from Proposition 1 that $\gcd(d''_1, d''_2) = 1$ and $d''_1 d''_2 = \frac{n^2+1}{2}$. But, $\gcd(d_1, d_2) = p d''_1 d''_2$ should be a divisor of $\frac{n^2+1}{2}$ which is not possible because $p > 1$. So, in this case we also conclude that $g = 1$.

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

and using $g = 1$, we obtain

$$(d_2 - d_1)^2 = (\delta n)^2 - 2 \frac{(n^2 + 1)}{d},$$

$$d(d_2 - d_1)^2 = \delta^2 n^2 d - 2n^2 - 2,$$

$$d(d_2 - d_1)^2 = (d\delta^2 - 2)n^2 - 2.$$

In the equation

$$(\delta^2 d - 2)n^2 - d(d_2 - d_1)^2 = 2,$$

we set $(d_2 - d_1) = 2y$ (number $d_2 - d_1$ is an even number because d_1, d_2 are odd integers), and we get

$$(\delta^2 d - 2)n^2 - 4dy^2 = 2.$$

If we divide both sides of the above equation by 2, then it becomes

$$(2d(2k + 1)^2 - 1)n^2 - 2dy^2 = 1.$$

Now, if we define $\delta' = \frac{\delta}{2} = 2k + 1$, we get

$$(2\delta'^2 d - 1)n^2 - 2dy^2 = 1. \quad (4.4)$$

We will prove by applying Lemma 1 that the above Pell equation (4.4) has no solutions.

To be able to apply Lemma 1, we have to deal with an equation of the form

$$x^2 - Dy^2 = 1.$$

We have $a = 2d\delta'^2 - 1$, $a > 1$ (because $\delta' \geq 3$) and $D = ab = 2d(2\delta'^2 d - 1)$ is not a perfect square because $2d(2\delta'^2 d - 1) \equiv 2 \pmod{4}$. We need to find the least positive integer solution of the equation

$$u^2 - 2d(2\delta'^2 d - 1)v^2 = 1. \quad (4.5)$$

For that purpose we find the continued fraction expansion of the number

$$\sqrt{2d(2\delta'^2 d - 1)}, \quad \delta' \geq 3.$$

We know that

$$\sqrt{2d(2\delta'^2 d - 1)} = [a_0; \overline{a_1, a_2, \dots, a_{l-1}, 2a_0}],$$

where we calculate numbers a_i recursively

$$a_i = \left\lfloor \frac{s_i + a_0}{t_i} \right\rfloor, \quad s_{i+1} = a_i t_i - s_i, \quad t_{i+1} = \frac{d - s_{i+1}^2}{t_i}.$$

In our case, we obtain

$$\begin{aligned} a_0 &= \lfloor \sqrt{2d(2\delta'^2 d - 1)} \rfloor = 2d\delta' - 1, \quad s_0 = 0, \quad t_0 = 1; \\ s_1 &= 2d\delta' - 1, \quad t_1 = 4d\delta' - 2d - 1, \quad a_1 = 1; \\ s_2 &= 2d\delta' - 2d, \quad t_2 = 2d, \quad a_2 = 2\delta' - 2; \\ s_3 &= 2d\delta' - 2d, \quad t_3 = 4d\delta' - 2d - 1, \quad a_3 = 1; \\ s_4 &= 2d\delta' - 1, \quad t_4 = 1, \quad a_4 = 2(2d\delta' - 1) = 2a_0. \end{aligned}$$

We get

$$\sqrt{2d(2\delta'^2 d - 1)} = [2d\delta' - 1; \overline{1, 2\delta' - 2, 1, 2(2d\delta' - 1)}].$$

Now, we can find the least positive integer solution of the equation (4.5). Because the length of the period of the expansion is $l = 4$, the least positive

integer solution of (4.5) is (p_3, q_3) , where numbers p_i, q_i , $i = 0, 1, 2, 3$ are calculated recursively

$$\begin{aligned} p_0 &= a_0, \quad p_1 = a_0 a_1 + 1, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}, \quad k = 2, 3. \end{aligned}$$

We obtain

$$\begin{aligned} (p_0, q_0) &= (2d\delta' - 1, 1), \quad (p_1, q_1) = (2d\delta', 1), \quad (p_2, q_2) = (4\delta'^2 d - 2d\delta' - 1, 2\delta' - 1), \\ (p_3, q_3) &= (4\delta'^2 d - 1, 2\delta'). \end{aligned}$$

So, the least positive integer solution is $(p_3, q_3) = (u_0, v_0) = (4\delta'^2 d - 1, 2\delta')$ and we apply Lemma 1.

In our case we have $a = 2\delta'^2 d - 1$, $b = 2d$. From Lemma 1 we get

$$(4\delta'^2 d - 2) | 4\delta'^2 d, \quad 4d | (4\delta'^2 d - 2).$$

We can easily see that $4d | (4\delta'^2 d - 2)$ if and only if $4d | 2$ which is not possible because $d \in \mathbb{N}$. So, the equation (4.4) has no solutions. We have proved that there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. \square

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